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A FORMULA SEPARATING TWO THEORIES

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Abstract

If theory T+$\neg\psi$ is incomplete, then there is a formula φ such that $\mathrm{T} \subsetneq \mathrm{T}+\varphi \subsetneq \mathrm{T}+\psi$. In special case ψ =Con(T), Rosser sentence of T is a separating formula.

0. Preliminaries

We consider only formulas in Arithmetic. A theory is a set of sentences that contains all of its logical consequences. T+φ denotes the least theory which contains T$\cup\{\varphi\}$ (T is a theory, φ is a sentence).

$\mathrm{T}\vdash\varphi$ means "φ is provable in T".

1. A formula separating two theories

Definition Let T, S be theories.

A sentence φ separates T from S $\overset{\text{def}}{\Leftrightarrow}$ $\mathrm{T} \subsetneq \mathrm{T}+\varphi \subsetneq \mathrm{S}$

Lemma 1 Let T be a theory, ψ be a sentence, and $\mathrm{T} \subsetneq \mathrm{T}+\psi$. The following conditions on φ are equivalent.

1. φ separates T from T+ψ.
2. (i) $\mathrm{T}\nvdash\varphi$, and

   (ii) $\mathrm{T}\vdash\psi\rightarrow\varphi$, $\mathrm{T}\nvdash\varphi\rightarrow\psi$.

Proof) $\mathrm{T}\subsetneq \mathrm{T}+\varphi$ is equivalent to (i). $\mathrm{T}+\varphi\subsetneq \mathrm{T}+\psi$ is equivalent to (ii). □

Remark $\mathrm{T} \subsetneq \mathrm{T}+\varphi \subsetneq \mathrm{T}+\psi \Leftrightarrow \mathrm{T}+\neg\psi \subsetneq \mathrm{T}+\neg\varphi \subsetneq$ Incon. (Incon is the inconsistent theory.)

Theorem If T+$\neg\psi$ is incomplete, then there is a formula φ separating T from T+ψ.
Proof) There is a sentence φ' such that T+$\neg\psi \nvdash \varphi'$ and T+$\neg\psi \nvdash \neg\varphi'$ because T+$\neg\psi$ is incomplete. Let $\varphi \equiv \varphi' \vee \psi$. It is easily verified that φ satisfy the conditions of the above remark. So this φ is a separating sentence T from T+ψ. □

Remark If T is a consistent primitive recursive theory, then T $\nvdash$ Con(T) by second incompleteness theorem [2]. So T+$\neg$Con(T) is a consistent primitive recursive theory, too. We have that T+$\neg$Con(T) is incomplete by Rosser's theorem [2]. So there is a theory between T and T+Con(T) by above theorem.

2. On a formula separating T from T+Con(T)

In the previous section, we showed that there is a theory between T and T+Con(T) (T is consistent and primitive recursive). Actually, let φ be a Rosser sentence of T+$\neg$Con(T), then $\neg\varphi \vee$Con(T) and $\varphi\vee$Con(T) are separating sentences. In this section, we show that Rosser sentence of T itself is a sentence separating T from T+Con(T). We will use certain results on the provability predicate. See [1], [2].

Definition
Rosser sentence φ of T is a sentence such that

$$\mathrm{T} \vdash \varphi \leftrightarrow \neg \mathrm{Pr}^*(\ulcorner \varphi \urcorner) \tag{1}$$

where $\mathrm{Pr}^*(x) \equiv \exists y(\mathrm{Prov_T}(x,y) \wedge \forall z \leq y(\neg \mathrm{Prov_T}(\mathrm{not}(x), z)))$.

Theorem Rosser sentence φ is a sentence separating T from T+Con(T).
Proof) We show (i) T$\nvdash\varphi$, (ii) T$\vdash$Con(T) $\to \varphi$, (iii) T$\nvdash \varphi \to$ Con(T), and then lemma 1 implies that φ is the separating sentence.
(i) T$\nvdash\varphi$ is well-known as Rosser's theorem.
(ii) By (1), we have

$$\mathrm{T}+\neg\varphi \vdash \exists y(\mathrm{Prov_T}(\ulcorner \varphi \urcorner, y)).$$

That is

$$\mathrm{T}+\neg\varphi \vdash \mathrm{Pr}(\ulcorner \varphi \urcorner). \tag{2}$$

On the other hand, we have

$$\mathrm{T} \vdash \mathrm{Con(T)} \to \neg\mathrm{Pr}(\ulcorner \varphi \urcorner). \tag{3}$$

$$\mathrm{T} \vdash \mathrm{Con(T)} \to \neg\mathrm{Pr}(\ulcorner \neg\varphi \urcorner). \tag{4}$$

by formalized Rosser's theorem. From (2) and (3), we have

$$\mathrm{T}+\neg\varphi \vdash \neg\mathrm{Con(T)},$$
$$\mathrm{T} \vdash \neg\varphi \to \neg\mathrm{Con(T)},$$
$$\mathrm{T} \vdash \mathrm{Con(T)} \to \varphi.$$

(iii) Assume that $\mathrm{T} \vdash \varphi \to \mathrm{Con(T)}$. By the derivability conditions of the provabiity predicate Pr(), we have

$$\mathrm{T} \vdash \mathrm{Pr}(\ulcorner \neg\mathrm{Con(T)} \to \neg\varphi \urcorner),$$

$$\mathrm{T} \vdash \mathrm{Pr}(\ulcorner \neg\mathrm{Con(T)} \urcorner) \to \mathrm{Pr}(\ulcorner \neg\varphi \urcorner). \tag{5}$$

By (4) and (5),

$$\mathrm{T} \vdash \mathrm{Pr}(\ulcorner \neg\mathrm{Con(T)} \urcorner) \to \neg\mathrm{Con(T)}.$$

Then we have

$$\mathrm{T} \vdash \neg\mathrm{Con(T)}$$

by Löb's theorem, and by assumption

$$\mathrm{T} \vdash \neg\varphi\,.$$

This contradicts that φ is Rosser sentence. □

References

[1] G. E. Boolos and R. C. Jeffery. *Computability and Logic 3rd ed.* . CAMBRIDGE UNIVERSITY PRESS 1989

[2] C. Smoryński. *The Incompleteness Theorems*, Handbook of Math. Logic (J. Barwise ed.) . North-Holland 1977. pp.821-865